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The Cauchy problem of Backward Stochastic Super-Parabolic Equations with Quadratic Growth

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Abstract The paper is devoted to the Cauchy problem of backward stochastic super-parabolic equations with quadratic growth. We prove two Itô formulas in the whole space. Furthermore, we prove the existence of weak solutions for the case of one-dimensional state space, and the uniqueness of weak solutions without constraint on the state space.

Keywords Backward stochastic differential equation · Quadratic growth · Weak solution · Super-parabolic · Itô's formula

AMS Subject Classification 60H15; 35R60; 93E20

1 Introduction

Let d and d_0 be integers and $\{W_t := (W_t^1, \ldots, W_t^{d_0})^*, 0 \leq t \leq T\}$ be a d_0-dimensional standard Brownian motion defined on some probability space $(\Omega, \mathscr{F}, P)$. Denote by $\{\mathscr{F}_t, 0 \leq t \leq T\}$ the augmented natural filtration of the standard Brownian motion W, and by $\mathscr{P}$ the predictable field with respect to the filtration $\{\mathscr{F}_t, 0 \leq t \leq T\}$.

Consider the Cauchy problem of the backward stochastic parabolic equation (BSPE):

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$$du = -\left[\sum_{i=1}^{d}\left(\sum_{j=1}^{d} a^{ij}u_{x^j} + \sum_{k=1}^{d_0}\sigma^{ik}q^k\right)_{x^i} + f(t,x,u,u_x,q)\right]dt$$
$$+\sum_{k=1}^{d_0} q^k dW_t^k, \quad (t,x)\in[0,T)\times\mathbb{R}^d \tag{1}$$

with the terminal condition

$$u(T,x)=\varphi(x), \quad x\in\mathbb{R}^d. \tag{2}$$

The BSPE is called super-parabolic if there are two positive constants μ_1 and μ_2 such that the coefficients a^{ij} and σ^{ik} satisfy the following condition:

$$\mu_1 I_d \le \left[2\left(a^{ij}\right) - \left(\sigma^{ik}\right)\left(\sigma^{jk}\right)^*\right](t,x) \le \mu_2 I_d, \tag{3}$$

with I_d being the identity matrix in $\mathbb{R}^d$. Assume that the BSPE is super-parabolic and that the generator $f(\omega,t,x,v,p,r)$ (the argument ω is usually omitted below) has the following quadratic growth:

$$|f(t,x,v,p,r)| \le \lambda_0(t,x) + \lambda_1|v| + \lambda_2(|p|^2+|r|^2), \tag{4}$$

where $\lambda_1 \ge 0$ and $\lambda_2 > 0$ are constants, and the predictable function $\lambda_0(t,x)$ has some integrability property (see Theorems 1 and 2 for more details).

When $\sigma \equiv 0$ and the data (f,ψ) is invariant with the space variable x, the preceding BSPE is reduced to a backward stochastic (ordinary) differential equation (BSDE), whose general form reads

$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s)\,ds - \int_t^T Z_s\,dW_s, \quad t\in[0,T]. \tag{5}$$

Here, the function $g:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^{d\times d_0}\to\mathbb{R}^d$ is called the generator of BSDE (5). The history of BSDE (5) can be traced to Bismut (1973) for the linear case, and to Bismut (1976) for a specifically structured matrix-valued nonlinear case where the matrix-valued generator contains a quadratic form of the second unknown. The uniformly Lipschitz case was later studied by Pardoux and Peng (1990).

Bismut (1976) derived a matrix-valued BSDE of a *quadratic* generator—the so-called backward stochastic Riccati equation (BSRE) in the study of linear quadratic optimal control with random coefficients, while he could not solve it in general. In that paper, he described the difficulty and failure of his fixed-point techniques in the proof of the existence and uniqueness for BSDE of a quadratic generator (i.e., the so-called quadratic BSDE). It has inspired subsequent intensive efforts in the research of quadratic BSDE (5). Nowadays, much progress has been made on this issue: Kobylanski (2000) and Briand and Hu (2006, 2008) gave the existence and uniqueness result for the case of a scalar-valued ($d = 1$) quadratic BSDE, Tang (2003, 2015) solved (using the stochastic maximum principle in Tang (2003) and dynamic programming in Tang (2015)) the existence and uniqueness result (posed by Bismut (1976)) for the general BSRE, and Tevzadze (2008) proved the existence and uniqueness result for a multi-dimensional quadratic BSDE (5) under the assumption that the terminal value is *sufficiently small* in the supremum norm (also called the *small terminal value problem*). Frei and dos Reis (2011) constructed a counterexample to show that a multi-dimensional quadratic BSDE (5) might fail to have a global solution (Y, Z) on $[0, T]$ such that Y is essentially bounded, which illustrates the difficulty of the quadratic part contributing to the underlying scalar generator as an unbounded process—the exponential of whose time integral is likely to have no finite expectation. Hu and Tang (2016) give the existence and uniqueness result for multi-dimensional BSDEs of diagonally quadratic generators (see El Karoui and Hamadène (2003) for a background of a diagonally quadratic system of BSDEs).

Backward stochastic partial differential equations (BSPDEs) have recently received a lot of attentions. The existence, uniqueness, and regularity of solutions to the Cauchy problem of BSPEs is fairly complete nowadays. See, among others, Du et al. (2012) for an L^p theory for non-degenerate BSPEs, Du and Zhang (2013) for the existence and uniqueness of degenerate parabolic BSPEs, and the relevant references therein. The previous research usually assumes that the generator f is uniformly Lipschitz in the unknown variables. Du and Chen (2012) study the Cauchy−Dirichlet problem of a super-parabolic quadratic BSPDE in a simply connected bounded domain $\mathcal{D}$:

$$du = -\left[(a^{ij}u_{x^j} + \sigma^{ik}q^k)_{x^i} + f(t, x, u, u_x, q)\right]dt + q^k dW_t^k,$$

with the terminal and boundary conditions:

$$\begin{cases} u(t, x) = 0, & t \in [0, T], x \in \partial\mathcal{D}, \\ u(T, x) = \varphi(x), & x \in \mathcal{D}, \end{cases}$$

and using the technique of exponential transformation developed by Kobylanski (2000), they prove the existence and uniqueness of weak solutions.

The paper considers the Cauchy problem of super-parabolic BSPDEs with quadratic growth in the second unknown variable. The Cauchy problem involves the whole spatial integrals, which might introduce some unbounded issue to the quadratic BSPDE and give rise to new difficulty. Two new Itô's formulas are proved for suitable functions defined in the whole space $\mathbb{R}^d$, which are crucial to establish the existence and uniqueness of weak solutions.

The Cauchy problem of (super-parabolic) BSPDEs with quadratic growth in the second unknown variable arises naturally in the solution of the risk-sensitive optimal control problem as the associated Hamilton−*Jacobi*−Bellman (HJB) equation. More precisely, consider the controlled non-Markovian stochastic differential equations:

$$X_t = x + \int_s^t b(r, X_r, v_r)\, dr + \int_s^t \sigma^k(r, X_r, v_r)\, dW_r^k, \quad t \in [s, \infty) \tag{6}$$

and the risk-sensitive cost functional:

$$J_{s,x}(v) = -\frac{1}{\mu}\mathbb{E}^{s,x}\left[\exp -\left(\mu\left(\int_s^T h(s, X_s, v_s)\, ds + \varphi(X_T)\right)\right)\right], \tag{7}$$

where the nonzero constant μ is the risk parameter of the controller, whose sign indicates the attitude (averse or preferable) to the risk . The control v takes values in a given set U and is required to satisfy some integrability property. The associated HJB equation reads

$$\begin{aligned} -dV(t,x) &= G\left(t,x,(V_x,V_{xx},q,q_x)(t,x)\right)\,dt - q^k(t,x)\,dW_t^k, \quad (t,x)\in[0,T)\times\mathbb{R}^d;\\ V(T,x) &= \varphi(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{8}$$

where the nonlinear partial differential operator G is defined for $(t,x)\in[0,T]\times\mathbb{R}^d$ and $(y,Y,z,Z)\in\mathbb{R}^d\times\mathbb{R}^{d\times d}\times\mathbb{R}^{d_0}\times\mathbb{R}^{d_0\times d}$,

$$\begin{aligned} &G\left(t,x,y,Y,z,Z\right)\\ =:&\inf_{v\in U}\left\{\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^*(t,x,v)Y+\sigma(t,x,v)Z\right)+\langle b(t,x,v),y\rangle\right.\\ &\left.+h(t,x,v)-\frac{1}{2}\mu\left|z+\sigma^*(t,x,v)y\right|^2\right\}. \end{aligned} \tag{9}$$

When the diffusion coefficient σ does not depend on the control variable, we have

$$\begin{aligned} &G\left(t,x,y,Y,z,Z\right)\\ =:&\frac{1}{2}\operatorname{Tr}\left(\sigma\sigma^*(t,x)Y+\sigma(t,x)Z\right)-\frac{1}{2}\mu\left|z+\sigma^*(t,x)y\right|^2\\ &+\inf_{v\in U}\left\{\langle b(t,x,v),y\rangle+h(t,x,v)\right\}, \end{aligned} \tag{10}$$

and the HJB Eq. (8) can be written into the form of BSPDE (1) when the coefficients σ is sufficiently smooth in the state x.

The rest of the paper is organized as follows. In Section 2, we introduce notations, definitions, and some lemmas. In Section 3, we first prove two Itô's formulas in the whole space, and then study the existence and uniqueness of a weak solution to the BSPE.

2 Preliminaries

2.1 Notation

Denote by v^i the i-th component of the vector $v\in\mathbb{R}^d$ for $i=1,2,\cdots,d$, and by a^{ij} the (i,j)-entry of the matrix $a\in\mathbb{R}^{m\times n}$ for $i=1,2,\cdots,m$ and $j=1,2,\cdots,n$. For a map u defined on the set $\Omega\times[0,T]\times\mathbb{R}^d$, the image $u(\omega,t,x)$ at the point (ω,t,x) is occasionally simplified as $u(t,x)$. Let u_{x^i} or D_iu be the partial differential of the function u with respect to x^i. We also use the convention that the repeated superscripts or subscripts imply the summation over the corresponding super- and sub-scripts.

For Banach space $\mathcal{B}$ and $p\in[1,+\infty]$, denote by $L^p_{\mathscr{P}}(\Omega\times[0,T],\ \mathcal{B})$ the Banach space of all L^p-integrable predictable processes $X:\Omega\times[0,T]\to\mathcal{B}$, equipped with the norm

$$\|X\|_{L^p_{\mathscr{P}}(\Omega\times[0,T],\ \mathcal{B})} := \left(\mathbb{E}\int_0^T\|X_t\|^p_{\mathcal{B}}\,dt\right)^{\frac{1}{p}};$$

by $C([0,T],\ \mathcal{B})$ the Banach space of all continuous maps $X:[0,T]\to\mathcal{B}$, equipped with the norm

$$\|X\|_{C([0,T],\ \mathcal{B})} := \sup_{t\in[0,T]} \|X_t\|_{\mathcal{B}};$$

and by $L^p(\Omega,\ \mathcal{B})$ the Banach space of all L^p-integrable maps $X:\Omega\to\mathcal{B}$, equipped with the norm

$$\|X\|_{L^p(\Omega,\ \mathcal{B})} = \left(\mathbb{E}\|X\|^p_{\mathcal{B}}\right)^{\frac{1}{p}}.$$

Let $L^p(E)$ be the Banach space of all real L^p-integrable functions f defined on the measure space $(E,\mathcal{E},\mu)$, equipped with the norm

$$\|f\|_{L^p(E)} := \left(\int_E |f|^p d\mu\right)^{\frac{1}{p}}.$$

For a subset $\mathcal{D}\subset\mathbb{R}^d$, denote by $H^m(\mathcal{D})$ or $H^m_0(\mathcal{D})$ the Sobolev space $W^{m,2}(\mathcal{D})$ or $W^{m,2}_0(\mathcal{D})$. Write $H^m(\mathbb{R}^d)$ for $H^m_0(\mathbb{R}^d)$. Moreover, denote by $\langle\cdot,\cdot\rangle^m_{\mathcal{D}}$ the inner product in the Hilbert space $H^m(\mathcal{D})$ or $H^m_0(\mathcal{D})$, and it is simplified by $\langle\cdot,\cdot\rangle_{\mathcal{D}}$ if $m=0$. Denote by $C^\infty_c(\mathcal{D})$ the totality of infinitely differentiable functions of compact supports in $\mathcal{D}$, and by $C^m_b(\mathcal{D})$ the totality of m-times differentiable functions, defined on $\mathcal{D}$, with all the partial differentials being bounded.

For $\mathcal{D}\subset\mathbb{R}^d$, define

$$\mathbb{L}^p(\mathcal{D}) := L^p_{\mathscr{P}}(\Omega\times[0,T],\ L^p(\mathcal{D})),\quad \mathbb{H}^{m-1}(\mathcal{D}) := L^2_{\mathscr{P}}(\Omega\times[0,T],\ H^{m-1}(\mathcal{D})),$$

$$\mathbb{H}^m_0(\mathcal{D}) := L^2_{\mathscr{P}}(\Omega\times[0,T],\ H^m_0(\mathcal{D})),\quad \mathbb{H}^m(\mathcal{D};\ \mathbb{R}^d) := \left(\mathbb{H}^m(\mathcal{D})\right)^d.$$

Write $\mathbb{H}^m(\mathbb{R}^d)$ for $\mathbb{H}^m_0(\mathbb{R}^d)$. Here, $m=1,2$.

Finally, for Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, $\mathcal{B}_1+\mathcal{B}_2$ stands for the space spanned by $\mathcal{B}_1$ and $\mathcal{B}_2$, that is, $\mathcal{B}_1+\mathcal{B}_2$ is the totality of all the sums x_1+x_2 with $x_1\in\mathcal{B}_1$ and $x_2\in\mathcal{B}_2$.

2.2 Definitions and lemmas

Consider the definition of a weak solution to the BSPE.

Definition 1 *A pair of random fields $(u,q)\in\mathbb{H}^1(\mathbb{R}^d)\times\mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$ is called a weak solution to BSPE (1)–(2) if $\forall\,\eta\in C^\infty_c(\mathbb{R}^d)$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^d} u(t,x)\eta(x)dx - \int_{\mathbb{R}^d}\varphi(x)\eta(x)dx \\
&= \int_t^T\int_{\mathbb{R}^d}\left[-(a^{ij}u_{x^j}+\sigma^{ik}q^k)\eta_{x^i} + f(s,x,u,u_x,q)\,\eta\right]dxds \qquad (11)\\
&\quad -\int_t^T\int_{\mathbb{R}^d} q^k\eta\, dxdW^k_s, \quad a.s.
\end{aligned}$$

A super-parabolic BSPE is defined as follows.

Definition 2 *BSPE (1)–(2) is called super-parabolic if there are two positive constants μ_1 and μ_2 such that the inequality (3) holds for all (ω,t,x).*

As an immediate consequence, we have:

Lemma 1 *Assume that the inequality (3) is satisfied. Then, there is a constant* $\mu_0 > 0$ *such that for any* $(p, r) \in \mathbb{R}^d \times \mathbb{R}^{d_0}$, *we have*

$$2a^{ij} p^i p^j + 2\sigma^{ik} p^i r^k + |r|^2 \geq \mu_0(|p|^2 + |r|^2). \tag{12}$$

The proof of the last lemma is referred to Du (2011, Lemma 4.3, page 46).

The following Itô's formula for functions defined in a simply connected bounded domain is a special case of Qiu and Tang (2012, Lemma 3.3, page 2445) when $f^0 \in \mathbb{L}^2(\mathcal{D})$.

Lemma 2 *Assume that* $\mathcal{D}$ *is a simply connected bounded domain of* $\mathbb{R}^d$, *and that* $(f^0, f^i, q^k) \in \mathbb{L}^1(\mathcal{D}) \times (\mathbb{H}^0(\mathcal{D}))^2$ *for* $i = 1, 2, \cdots, d$ *and* $k = 1, 2, \cdots, d_0$, *and* $u \in \mathbb{H}^1_0(\mathcal{D}) \cap L^2(\Omega,\ C([0, T],\ L^2(\mathcal{D})))$. *Further, for any* $\eta \in C_c^\infty(\mathbb{R}^d)$, *we have for any* $t \in [0, T]$,

$$\begin{aligned}
&\langle u(t, \cdot), \eta\rangle_{\mathcal{D}} - \langle u(T, \cdot), \eta\rangle_{\mathcal{D}} \\
&= \int_t^T \Big[\langle f^0, \eta\rangle_{\mathcal{D}} - \langle f^i, \eta_{x^i}\rangle_{\mathcal{D}}\Big] ds - \int_t^T \langle q^k(s, \cdot), \eta\rangle_{\mathcal{D}} dW_s^k, \quad a.s.
\end{aligned}$$

Then, for any twice differentiable function Φ *such that the first-order and second-order derivatives* Φ' *and* Φ'' *are bounded with* $\Phi'(0) = 0$, *we have*

$$\begin{aligned}
&\int_{\mathcal{D}} \Phi(u(t, x))dx - \int_{\mathcal{D}} \Phi(u(T, x))dx \\
&= \int_t^T \int_{\mathcal{D}} \left[\Phi'(u) f^0 - \Phi''(u) u_{x^i} f^i - \frac{1}{2}\Phi''(u)|q|^2\right](s, x) dx ds \\
&\quad - \int_t^T \int_{\mathcal{D}} \Phi'(u(s, x)) q^k(s, x) dx dW_s^k, \quad a.s.
\end{aligned} \tag{13}$$

The lemma in the general case of $f^0 \in \mathbb{L}^1(\mathcal{D})$ can be proved via approximating $f^0 \in \mathbb{L}^1(\mathcal{D})$ with a sequence of fields in the space $\mathbb{L}^2(\mathcal{D})$.

The following lemma generalizes that of Lepeltier and San Martin (1997) and Kobylanski (2000, Lemma 2.5), and the proof is the same as theirs.

Lemma 3 *Let the sequence* $\{X_n\}_n$ *converge to* X *strongly in* $\mathbb{H}^0(\mathbb{R}^d)$. *Then, there is a subsequence* $\{n_i\}_i$ *such that* X_{n_i} *converges to* X *almost surely, and* $\widetilde{X} := \sup_i |X_{n_i}| \in \mathbb{H}^0(\mathbb{R}^d)$.

Lemma 4 *Assume that* $\{W_t, 0 \leq t \leq T\}$ *is a* d_0*-dimensional standard Brownian motion, and* $\varphi : [0, T] \times \Omega \to \mathbb{R}^{d_0}$ *is* $\mathscr{F}_t$*-adapted and* $\int_0^T |\varphi_t|^2 dt < +\infty$ *almost surely. If* $\mathbb{E}\sqrt{\int_0^T |\varphi_t|^2 dt} < +\infty$, *the local martingale* $\left\{\int_0^t \varphi_s^k dW_s^k, 0 \leq t \leq T\right\}$ *is uniformly integrable.*

3 Existence and uniqueness of weak solutions

3.1 Main Results

Consider the following five assumptions.

($\mathbf{H}1$) The function $f(\cdot,\cdot,\cdot,v,p,r)$ is $\mathscr{P}\times\mathscr{B}(\mathbb{R}^d)$-measurable for each $(v,p,r)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d_0}$, and $f(\omega,t,x,\cdot,\cdot,\cdot)$ is continuous for each $(\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}^d$.

($\mathbf{H}2$) There are a nonnegative random function $\lambda_0\in\mathbb{L}^\infty(\mathbb{R}^d)\cap\mathbb{L}^2(\mathbb{R}^d)$ and constants $\lambda_1>0$ and $\lambda_2>0$, such that the generator f has the following condition of quadratic growth:

$$|f(t,x,v,p,r)|\le\lambda_0(t,x)+\lambda_1|v|+\lambda_2(|p|^2+|r|^2). \tag{14}$$

($\mathbf{H}3$) The coefficients a^{ij} and σ^{ik} are $\mathscr{P}\times\mathscr{B}(\mathbb{R}^d)$-measurable and bounded, $a^{ij}=a^{ji}$, and satisfy the inequality (3) (also called the super-parabolic condition).

($\mathbf{H}4$) The nonzero terminal value $\varphi:\Omega\times\mathbb{R}^d\to\mathbb{R}$ is $\mathscr{F}_T\times\mathscr{B}(\mathbb{R}^d)$-measurable, and $\varphi\in L^\infty(\Omega\times\mathbb{R}^d)\cap L^2(\Omega\times\mathbb{R}^d)$.

($\mathbf{H}5$) The space dimension $d=1$.

Remark 1 *In the inequality (14) (see* ($\mathbf{H}2$)*), for simplicity, without loss of generality we might take $\lambda:=\frac{\lambda_2}{\mu_0}$, where μ_0 comes from (12). Subsequently, unless stated otherwise, quadratic growth refers to:*

$$|f(t,x,v,p,r)|\le\lambda_0(t,x)+\lambda_1|v|+\lambda\mu_0(|p|^2+|r|^2). \tag{15}$$

We have the following existence of weak solutions.

Theorem 1 *Let the five assumptions* ($\mathbf{H}1$)$-$($\mathbf{H}5$) *be satisfied. Then, BSPE (1)–(2) has a weak solution $(u,q)\in\mathbb{H}^1(\mathbb{R}^d)\times\mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$ with $u\in\mathbb{L}^\infty(\mathbb{R})$.*

The proof of the theorem is divided into three steps. Step one is devoted to the a priori estimate of the weak solution (see Lemma 5). In this step, we make the exponential transformation, and use a suitable Itô's formula which has to be proved here in full details. Step 2 is devoted to the proof of the monotone convergence theorem in the case of quadratic generators (see Lemma 6). In Step 3, Theorem 1 on the existence of weak solutions to BSPEs is proved via the exponential transformation.

Noting that the five assumptions of Theorem 1 do not assume any Hölder continuity of the function f even in the first unknown variable (that is, in the fourth argument), we do not expect any uniqueness of the weak solution without imposing extra conditions. In fact, Fujita (1969, Theorem 3.1, page 111) is a nonuniqueness theorem, and includes the following example: BSPE (1)-(2) has at least two solutions $(0,0)$ and $(\hat u,0)$ with $\hat u(t,x):=(\alpha(T-t))^{\frac{1}{\alpha}}$, $(t,x)\in[0,T]\times\mathbb{R}^d$ when $\varphi\equiv0$, $(a_{ij})\equiv I_d$, $\sigma\equiv0$, and $f=v^{1-\alpha}$ with $\alpha\in(0,1)$. To address the uniqueness of weak solutions, we consider the following two assumptions.

($\mathbf{H}6$) There are a deterministic function $l_0\in L^1([0,T]\times\mathbb{R}^d)\cap L^\infty([0,T]\times\mathbb{R}^d)$, $l_2(t)\in L^2([0,T])$ and a constant $l_1>0$ such that the generator f has the following quadratic growth

$$|f(t,x,v,p,r)| \le l_0(t,x) + l_1(|p|^2 + |r|^2) \tag{16}$$

and is differentiable in (p, r) with the partial derivatives growing in a linear manner:

$$|f_p(t,x,v,p,r)| + |f_r(t,x,v,p,r)| \le l_2(t) + l_1(|p| + |r|).$$

(**H**7) $\forall\, \varepsilon > 0$, there is a function $\Lambda_\varepsilon(\cdot) \in L^1([0,T])$ such that the generator f satisfies the following

$$|f_v(t,x,v,p,r)| \le \Lambda_\varepsilon(t) + \varepsilon(|p|^2 + |r|^2).$$

We have the following result on uniqueness of weak solutions.

Theorem 2 *Let the four assumptions* (**H**1)*,* (**H**3)*,* (**H**6)*, and* (**H**7) *be satisfied. Assume that for* $i = 1, 2$*, the pair* (u^i, q^i) *is a weak solution of BSPE (1)–(2) with* u^i *being bounded. Then, we have* $u^1 = u^2$.

Combining both Theorems 1 and 2, we have the following.

Theorem 3 *Let the seven assumptions* (**H**1) − (**H**7) *be satisfied. Then, BSPE (1)–(2) has a unique weak solution* $(u, q) \in \mathbb{H}^1(\mathbb{R}^d) \times \mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$ *with* $u \in \mathbb{L}^\infty(\mathbb{R})$.

3.2 Itô's formula for functions defined in the whole space $\mathbb{R}^d$

In this subsection, we prove two types of Itô's formulas, which are the key to our subsequent proof of the existence and uniqueness of a weak solution.

First, we have the following extension of Lemma 2 from simply connected bounded domains to the whole space $\mathbb{R}^d$.

Theorem 4 *Assume that random functions* $f^i, q^k \in \mathbb{H}^0(\mathbb{R}^d)$*,* $f^0 \in \mathbb{H}^0(\mathbb{R}^d) + \mathbb{L}^1(\mathbb{R}^d)$*, and* $u \in \mathbb{H}^1(\mathbb{R}^d) \cap L^2(\Omega,\ C([0,T],\ L^2(\mathbb{R}^d)))$*. For* $\forall\, \eta \in C_c^\infty(\mathbb{R}^d)$*, we have*

$$\begin{aligned}&\langle u(t,\cdot), \eta\rangle_{\mathbb{R}^d} - \langle u(T,\cdot), \eta\rangle_{\mathbb{R}^d}\\ &= \int_t^T \left[\langle f^0, \eta\rangle_{\mathbb{R}^d} - \langle f^i, \eta_{x^i}\rangle_{\mathbb{R}^d}\right] ds - \int_t^T \langle q^k, \eta\rangle_{\mathbb{R}^d} dW_s^k, \quad a.s.\end{aligned} \tag{17}$$

Then, for any $\Phi \in C_b^2(\mathbb{R})$ *such that* $\Phi'(0) = \Phi(0) = 0$*, we have*

$$\begin{aligned}&\int_{\mathbb{R}^d} \Phi(u(t,x))dx - \int_{\mathbb{R}^d} \Phi(u(T,x))dx\\ &= \int_t^T \int_{\mathbb{R}^d} \left[\Phi'(u) f^0 - \Phi''(u) u_{x^i} f^i - \frac{1}{2}\Phi''(u)|q|^2\right](s,x)dxds\\ &\quad - \int_t^T \int_{\mathbb{R}^d} \Phi'(u(s,x)) q^k(s,x) dx dW_s^k, \quad a.s.\end{aligned} \tag{18}$$

Proof We use the technique of mollifier to construct $\xi \in C_c^\infty(\mathbb{R}^d)$, such that

$$\mathrm{supp}(\xi) \subset \{|x| \le 1\}, \quad \int_{\mathbb{R}^d} \xi(x)dx = 1.$$

Then, we construct the truncating function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\psi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2. \end{cases}$$

Define $\xi^\varepsilon(x) := \varepsilon^{-d}\xi(\frac{x}{\varepsilon})$. For an arbitrary function $p(\cdot)$, define

$$p^\varepsilon(x) := \psi(\varepsilon x)\int_{\mathbb{R}^d} p(y)\xi^\varepsilon(x-y)dy, \quad x \in \mathbb{R}^d.$$

For fixed $x \in \mathbb{R}^d$, define $\eta(y) := \xi^\varepsilon(x-y)\psi(\varepsilon x)$. Putting it into (17), we have

$$\begin{aligned} & u^\varepsilon(t,x) - u^\varepsilon(T,x) \\ &= \int_t^T \left[f^{0,\varepsilon}(s,x) + D_i(f^{i,\varepsilon}(s,x)) \right] ds - \int_t^T q^{k,\varepsilon}(s,x)dW_s^k. \end{aligned} \tag{19}$$

Note that the supports of the mollified functions u^ε, q^ε, $f^{0,\varepsilon}$, and $f^{i,\varepsilon}$ are compact and included in the bounded domain

$$\mathcal{D}^\varepsilon := \left\{ x \;\middle|\; |x| \le \frac{2}{\varepsilon} \right\}.$$

On both sides of Eq. (19), multiplying by the function $\eta \in C_c^\infty(\mathbb{R}^d)$ and then integrating over $x \in \mathcal{D}^\varepsilon$, we have

$$\begin{aligned} & \langle u^\varepsilon(t,\cdot), \eta\rangle_{\mathcal{D}^\varepsilon} - \langle u^\varepsilon(T,\cdot), \eta\rangle_{\mathcal{D}^\varepsilon} \\ &= \int_t^T \left[\langle f^{0,\varepsilon}, \eta\rangle_{\mathcal{D}^\varepsilon} - \langle f^{i,\varepsilon}, \eta_{x^i}\rangle_{\mathcal{D}^\varepsilon} \right] ds - \int_t^T \langle q^{k,\varepsilon}, \eta\rangle_{\mathcal{D}^\varepsilon} dW_s^k, \quad a.s. \end{aligned} \tag{20}$$

Since $\Phi'(0) = 0$, applying Lemma 2 to (20), we have

$$\begin{aligned} & \int_{\mathcal{D}^\varepsilon} \Phi(u^\varepsilon(t,x))dx - \int_{\mathcal{D}^\varepsilon} \Phi(u^\varepsilon(T,x))dx \\ &= \int_t^T \int_{\mathcal{D}^\varepsilon} \left[\Phi'(u^\varepsilon) f^{0,\varepsilon} - \Phi''(u^\varepsilon) u^\varepsilon_{x^i} f^{i,\varepsilon} - \frac{1}{2}\Phi''(u^\varepsilon)|q^\varepsilon|^2 \right](s,x)dxds \\ &\quad - \int_t^T \int_{\mathcal{D}^\varepsilon} \Phi'(u^\varepsilon(s,x)) q^{k,\varepsilon}(s,x)dxdW_s^k, \quad a.s. \end{aligned} \tag{21}$$

In view of $\Phi(0) = 0$, we have

$$\int_{\mathbb{R}^d \setminus \mathcal{D}^\varepsilon} \Phi(u^\varepsilon(t,x))dx = \int_{\mathbb{R}^d \setminus \mathcal{D}^\varepsilon} \Phi(0)dx = 0.$$

Since q^ε, $f^{0,\varepsilon}$, and $f^{i,\varepsilon}$ vanish on the set $\mathbb{R}^d \setminus \mathcal{D}^\varepsilon$, the spatial integrals in (21) over the domain $\mathcal{D}^\varepsilon$ may be written into those over the whole space $\mathbb{R}^d$, that is, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Phi(u^\varepsilon(t,x))dx - \int_{\mathbb{R}^d} \Phi(u^\varepsilon(T,x))dx \\
&= \int_t^T \int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon) f^{0,\varepsilon} - \Phi''(u^\varepsilon) u^\varepsilon_{x^i} f^{i,\varepsilon} - \frac{1}{2}\Phi''(u^\varepsilon)|q^\varepsilon|^2 \right](s,x)dxds \qquad (22) \\
&\quad - \int_t^T \int_{\mathbb{R}^d} \Phi'(u^\varepsilon(s,x)) q^{k,\varepsilon}(s,x)dxdW^k_s.
\end{aligned}$$

It remains to prove that all the terms in the equality (22) for a common subsequence almost surely converge to their counterparts in equality (18).

In view of the properties of mollifier and truncation, for $\forall\ (t,\omega)$, $k = 1,2,\cdots,d_0,\ i = 1,2,\cdots,d$, we have

$$\begin{aligned}
&u^\varepsilon(t,\cdot) \xrightarrow{H^1(\mathbb{R}^d)} u(t,\cdot), \qquad \|u^\varepsilon(t,\cdot)\|_{H^1(\mathbb{R}^d)} \le \|u(t,\cdot)\|_{H^1(\mathbb{R}^d)}, \\
&q^{k,\varepsilon}(t,\cdot) \xrightarrow{H^0(\mathbb{R}^d)} q^k(t,\cdot), \quad \|q^{k,\varepsilon}(t,\cdot)\|_{H^0(\mathbb{R}^d)} \le \|q^k(t,\cdot)\|_{H^0(\mathbb{R}^d)}, \\
&f^{i,\varepsilon}(t,\cdot) \xrightarrow{H^0(\mathbb{R}^d)} f^i(t,\cdot), \quad \|f^{i,\varepsilon}(t,\cdot)\|_{H^0(\mathbb{R}^d)} \le \|f^i(t,\cdot)\|_{H^0(\mathbb{R}^d)}.
\end{aligned}$$

Since (18) and (22) are linear in f^0, it is sufficient to consider both cases of $f^0 \in \mathbb{H}^0\left(\mathbb{R}^d\right)$ and $f^0 \in \mathbb{L}^1(\mathbb{R}^d)$. Identically as before, we see that $\forall\ (t,\omega)$,

$$f^{0,\varepsilon}(t,\cdot) \xrightarrow{H^0(\mathbb{R}^d)} f^0(t,\cdot), \quad \|f^{0,\varepsilon}(t,\cdot)\|_{H^0(\mathbb{R}^d)} \le \|f^0(t,\cdot)\|_{H^0(\mathbb{R}^d)}$$

for $f^0 \in \mathbb{H}^0\left(\mathbb{R}^d\right)$; and

$$f^{0,\varepsilon}(t,\cdot) \xrightarrow{L^1(\mathbb{R}^d)} f^0(t,\cdot), \quad \|f^{0,\varepsilon}(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|f^0(t,\cdot)\|_{L^1(\mathbb{R}^d)}$$

for $f^0 \in \mathbb{L}^1\left(\mathbb{R}^d\right)$. For the detailed proof, see Chen (2005, Theorem 1.1).

On the other hand, in view of Lemma 3, we see that $\forall\ (t,\omega)$, there is a subsequence of $\{u^\varepsilon\}$, still denoted without loss of generality by $\{u^\varepsilon\}$, such that $\sup_\varepsilon |u^\varepsilon(t,\cdot)| \in H^1(\mathbb{R}^d)$ and $u^\varepsilon(t,\cdot) \xrightarrow{a.e.} u(t,\cdot)$.

Finally, assume that $|\Phi| + |\Phi'| + |\Phi''| \le M$. In what follows, we show the convergence of each term.

First, we show the following convergence: for $\forall\ t \in [0,T]$,

$$\int_{\mathbb{R}^d} \Phi(u^\varepsilon(t,x))dx \xrightarrow{a.s.} \int_{\mathbb{R}^d} \Phi(u(t,x))dx. \qquad (23)$$

Since $\Phi'(0) = \Phi(0) = 0$, using Taylor's extension, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} \Phi(u^\varepsilon(t,x))dx - \int_{\mathbb{R}^d} \Phi(u(t,x))dx\right|\\ \leq& M\int_{\mathbb{R}^d}\left[|u^\varepsilon(t,x)|^2 + |u(t,x)|^2\right]dx\\ \leq& 2M\int_{\mathbb{R}^d}\left(\sup_\varepsilon |u^\varepsilon(t,x)|\right)^2 dx.\end{aligned}$$

In view of the convergence $u^\varepsilon(t,x) \xrightarrow{a.e.} u(t,x)$, we have

$$\Phi(u^\varepsilon(t,x)) \xrightarrow{a.e.} \Phi(u(t,x)).$$

Applying the dominated convergence theorem, we have (23).

Next, we show the following convergence

$$\int_t^T \int_{\mathbb{R}^d} \Phi'(u^\varepsilon) f^{0,\varepsilon} dxds \xrightarrow{a.s.} \int_t^T \int_{\mathbb{R}^d} \Phi'(u) f^0 dxds. \tag{24}$$

The proof is divided into both case of $f^0 \in \mathbb{L}^1(\mathbb{R}^d)$ and $f^0 \in \mathbb{H}^0(\mathbb{R}^d)$.

Case I. For $f^0 \in \mathbb{L}^1(\mathbb{R}^d)$, we have

$$\begin{aligned}&\left|\int_t^T \int_{\mathbb{R}^d}\left[\Phi'(u)f^0 - \Phi'(u^\varepsilon)f^{0,\varepsilon}\right]dxds\right|\\ \leq& \int_t^T \int_{\mathbb{R}^d}\left[|\Phi'(u^\varepsilon)||f^0 - f^{0,\varepsilon}| + |f^0||\Phi'(u^\varepsilon) - \Phi'(u)|\right]dxds\\ \leq& M\int_t^T \int_{\mathbb{R}^d}|f^0 - f^{0,\varepsilon}|dxds + \int_t^T \int_{\mathbb{R}^d}|f^0||\Phi'(u^\varepsilon) - \Phi'(u)|dxds.\end{aligned} \tag{25}$$

Since

$$\int_{\mathbb{R}^d}|f^0(s,x) - f^{0,\varepsilon}(s,x)|dx \to 0, \quad \forall\, s \in [t,T]$$

and

$$\int_{\mathbb{R}^d}|f^0(s,x) - f^{0,\varepsilon}(s,x)|dx \leq 2\|f^0\|_{L^1(\mathbb{R}^d)},$$

using the dominated convergence theorem, we see that the first term in (25) converges to 0. Since

$$\int_{\mathbb{R}^d}|f^0||\Phi'(u^\varepsilon) - \Phi'(u)|dx \leq 2M\int_{\mathbb{R}^d}|f^0|dx$$

and the right side of the last inequality is integrable in s on $[t,T]$, applying the dominated convergence theorem, we see that the second term of (25) and thus (25) $a.s.$ converges to 0.

Case II. For $f^0 \in \mathbb{H}^0(\mathbb{R}^d)$, in view of the first-order extension of Φ', we have

$$\begin{aligned}&\left|\int_t^T \int_{\mathbb{R}^d}\left[\Phi'(u)f^0 - \Phi'(u^\varepsilon)f^{0,\varepsilon}\right]dxds\right|\\ \leq& M\int_t^T \int_{\mathbb{R}^d}\left(|f^0 - f^{0,\varepsilon}||u^\varepsilon| + |f^0||u^\varepsilon - u|\right)dxds.\end{aligned} \tag{26}$$

Using Hölder's inequality, we have $\forall s \in [t, T]$

$$\begin{aligned}&\int_{\mathbb{R}^d} \Big(|f^0 - f^{0,\varepsilon}||u^\varepsilon| + |f^0||u^\varepsilon - u|\Big)\, dx\\ \leq &\| f^0(s,\cdot) - f^{0,\varepsilon}(s,\cdot)\|_{H^0(\mathbb{R}^d)}\ \|u^\varepsilon(s,\cdot)\|_{H^0(\mathbb{R}^d)}\\ &+ \| f^0(s,\cdot)\|_{H^0(\mathbb{R}^d)}\ \|u^\varepsilon(s,\cdot) - u(s,\cdot)\|_{H^0(\mathbb{R}^d)}\\ \leq &4\| f^0(s,\cdot)\|_{H^0(\mathbb{R}^d)}\ \|u(s,\cdot)\|_{H^0(\mathbb{R}^d)}.\end{aligned}$$

From the first inequality, we see the convergence to 0 of the following spatial integral

$$\int_{\mathbb{R}^d} \Big(|f^0 - f^{0,\varepsilon}||u^\varepsilon| + |f^0||u^\varepsilon - u|\Big)\, dx.$$

From the second inequality, we see that the preceding integral is dominated by the integral function $4\| f^0(s,\cdot)\|_{H^0(\mathbb{R}^d)}\|u(s,\cdot)\|_{H^0(\mathbb{R}^d)}, \quad s \in [0, T]$. Applying the dominated convergence theorem, we see that (26) $a.s.$ converges to 0. In conclusion, we have shown (24).

Now, we prove the following convergence

$$\int_t^T \int_{\mathbb{R}^d} \Phi''(u^\varepsilon) u^\varepsilon_{x^i} f^{i,\varepsilon} dxds \xrightarrow{a.s.} \int_t^T \int_{\mathbb{R}^d} \Phi''(u) u_{x^i} f^i dxds. \tag{27}$$

For $\forall\ i = 1, 2, \cdots, d$, we have

$$\begin{aligned}&\left|\int_t^T \int_{\mathbb{R}^d} \Big[\Phi''(u) u_{x^i} f^i - \Phi''(u^\varepsilon) u^\varepsilon_{x^i} f^{i,\varepsilon}\Big] dxds\right|\\ \leq &\int_t^T \int_{\mathbb{R}^d} |\Phi''(u^\varepsilon)||u_{x^i} f^i - u^\varepsilon_{x^i} f^{i,\varepsilon}| dxds + \int_t^T \int_{\mathbb{R}^d} |u_{x^i} f^i||\Phi''(u^\varepsilon) - \Phi''(u)| dxds\\ \leq &M \int_t^T \int_{\mathbb{R}^d} |u_{x^i} f^i - u^\varepsilon_{x^i} f^{i,\varepsilon}| dxds + \int_t^T \int_{\mathbb{R}^d} |u_{x^i} f^i||\Phi''(u^\varepsilon) - \Phi''(u)| dxds\\ \leq &M \int_t^T \int_{\mathbb{R}^d} \Big(|u^\varepsilon_{x^i}||f^i - f^{i,\varepsilon}| + |f^i||u_{x^i} - u^\varepsilon_{x^i}|\Big) dxds\\ &+ \int_t^T \int_{\mathbb{R}^d} |u_{x^i} f^i||\Phi''(u^\varepsilon) - \Phi''(u)| dxds.\end{aligned} \tag{28}$$

First, the first integral in the right side of the last inequality converges to 0, whose proof is identical to that of (26). Next, since

$$\int_{\mathbb{R}^d} |u_{x^i} f^i||\Phi''(u^\varepsilon) - \Phi''(u)| dx \leq \int_{\mathbb{R}^d} 2M |u_{x^i} f^i| dx$$

and the right side of the last inequality is integrable in s over $[t, T]$, applying the dominated convergence theorem, we see that the second integral in the right side of inequality (28) also converges to 0. Therefore, we have (27).

Now, we prove the following convergence:

$$\int_t^T \int_{\mathbb{R}^d} \Phi''(u^\varepsilon)|q^\varepsilon|^2 dxds \xrightarrow{a.s.} \int_t^T \int_{\mathbb{R}^d} \Phi''(u)|q|^2 dxds. \tag{29}$$

We have

$$
\begin{aligned}
&\left|\int_t^T \int_{\mathbb{R}^d} \left[\Phi''(u)|q|^2 - \Phi''(u^\varepsilon)|q^\varepsilon|^2\right] dxds\right| \\
&\leq \int_t^T \int_{\mathbb{R}^d} |\Phi''(u^\varepsilon)| \left||q|^2 - |q^\varepsilon|^2\right| dxds + \int_t^T \int_{\mathbb{R}^d} |q|^2 |\Phi''(u^\varepsilon) - \Phi''(u)| dxds \\
&\leq M \int_t^T \int_{\mathbb{R}^d} \left||q|^2 - |q^\varepsilon|^2\right| dxds + \int_t^T \int_{\mathbb{R}^d} |q|^2 |\Phi''(u^\varepsilon) - \Phi''(u)| dxds.
\end{aligned} \tag{30}
$$

Since the spatial integral

$$\int_{\mathbb{R}^d} \left||q(s,\cdot)|^2 - |q^\varepsilon(s,\cdot)|^2\right| dx \to 0, \quad \forall\, s \in [t,T],$$

and is dominated by $2\|q(t,\cdot)\|^2_{(H^0(\mathbb{R}^d))^{d_0}}$, and

$$\int_{\mathbb{R}^d} |q|^2 |\Phi''(u^\varepsilon) - \Phi''(u)| dx \leq 2M \int_{\mathbb{R}^d} |q|^2 dx$$

and the right side of the last inequality is integrable in s over $[t,T]$, applying the dominated convergence theorem, we see that both terms in inequality (30) converge to 0. Therefore, we have (29).

Finally, we prove the zero convergence and the martingale property of the term

$$\int_t^T \int_{\mathbb{R}^d} \Phi'(u^\varepsilon) q^{k,\varepsilon} dx dW_s^k. \tag{31}$$

Since $u \in L^2(\Omega,\ C([0,T],\ L^2(\mathbb{R}^d)))$, there is a constant $C > 0$ such that

$$\mathbb{E} \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq C.$$

Proceeding identically as before, for $k = 1, 2, \cdots, d_0$, we have

$$
\begin{aligned}
&\left\{\int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon(s,x)) q^{k,\varepsilon}(s,x) - \Phi'(u(s,x)) q^k(s,x)\right] dx\right\}^2 \\
\leq\ & 16M^2\ \|u(s,\cdot)\|^2_{H^0(\mathbb{R}^d)}\ \|q^k(s,\cdot)\|^2_{H^0(\mathbb{R}^d)}.
\end{aligned}
$$

While

$$\|u(s,\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq K(\omega), \quad a.s.,$$

where $K(\omega)$ only depends on ω. Since

$$
\begin{aligned}
&\int_t^T \left\{\int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon) q^{k,\varepsilon} - \Phi'(u) q^k\right] dx\right\}^2 ds \\
&\leq 16M^2 K(\omega) \int_t^T \|q^k(s,\cdot)\|^2_{H^0(\mathbb{R}^d)} ds,
\end{aligned}
$$

proceeding identically as before, for $\forall\, k$, we have

$$\left\{\int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon)) q^{k,\varepsilon} - \Phi'(u) q^k\right] dx\right\}^2 \to 0.$$

Using the dominated convergence theorem, we have

$$\sqrt{\int_t^T \left\{\int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon)q^{k,\varepsilon} - \Phi'(u)q^k\right] dx\right\}^2 ds} \xrightarrow{a.s.} 0. \tag{32}$$

On the other hand, in view of Hölder's inequality, we have

$$\begin{aligned}
&\sqrt{\int_t^T \left\{\int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon)q^{k,\varepsilon} - \Phi'(u)q^k\right] dx\right\}^2 ds} \\
\leq &\sqrt{16M^2 \int_t^T \|u(s,\cdot)\|^2_{H^0(\mathbb{R}^d)} \, \|q^k(s,\cdot)\|^2_{H^0(\mathbb{R}^d)} ds} \\
\leq &\sqrt{16M^2 \int_t^T \left[\sup_{s\in[0,T]} \|u(s,\cdot)\|^2_{H^0(\mathbb{R}^d)}\right] \|q^k(s,\cdot)\|^2_{H^0(\mathbb{R}^d)} ds} \\
\leq &\left(16M^2 \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{H^0(\mathbb{R}^d)}\right)^{\frac{1}{2}} \left(\int_t^T \|q^k(s,\cdot)\|^2_{H^0(\mathbb{R}^d)} ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{E}\sqrt{\int_t^T \left\{\int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon)q^{k,\varepsilon} - \Phi'(u)q^k\right] dx\right\}^2 ds} \\
\leq &\mathbb{E}\left[\left(16M^2 \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{H^0(\mathbb{R}^d)}\right)^{\frac{1}{2}} \left(\int_t^T \|q^k(s,\cdot)\|^2_{H^0(\mathbb{R}^d)} ds\right)^{\frac{1}{2}}\right] \\
\leq &\left(16M^2 \mathbb{E} \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{H^0(\mathbb{R}^d)}\right)^{\frac{1}{2}} \left(\mathbb{E}\int_t^T \|q^k(s,\cdot)\|^2_{H^0(\mathbb{R}^d)} ds\right)^{\frac{1}{2}} \\
\leq &\sqrt{16M^2C} \left(\mathbb{E}\int_t^T \|q^k(s,\cdot)\|^2_{H^0(\mathbb{R}^d)} ds\right)^{\frac{1}{2}},
\end{aligned}$$

and the random variable

$$\sqrt{\int_t^T \left\{\int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon)q^{k,\varepsilon} - \Phi'(u)q^k\right] dx\right\}^2 ds}$$

is dominated by the random variable

$$\left(4M \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{H^0(\mathbb{R}^d)}\right)^{\frac{1}{2}} \left(\int_t^T \|q^k(s,\cdot)\|^2_{H^0(\mathbb{R}^d)} ds\right)^{\frac{1}{2}}.$$

Combining (32), using the dominated convergence theorem and BDG inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left|\int_t^T \int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon)q^{k,\varepsilon} - \Phi'(u)q^k\right] dxdW_s^k\right| \\
&\leq \mathbb{E}\sqrt{\int_t^T \left\{\int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon)q^{k,\varepsilon} - \Phi'(u)q^k\right] dx\right\}^2 ds} \to 0.
\end{aligned} \tag{33}
$$

Hence, the sequence

$$
\int_t^T \int_{\mathbb{R}^d} \left[\Phi'(u^\varepsilon)q^{k,\varepsilon} - \Phi'(u)q^k\right] dxdW_s^k
$$

has a subsequence which almost surely converges to 0. Meanwhile, we have

$$
\mathbb{E}\sqrt{\int_0^T \left[\int_{\mathbb{R}^d} \Phi'(u)q^k dx\right]^2 ds} < +\infty.
$$

It follows from Lemma 4 that the process

$$
\int_0^t \int_{\mathbb{R}^d} \Phi'(u)q^k dxdW_s^k, \quad t \in [0, T]
$$

is a martingale, and has zero expectation. Now, we arrive at the convergence (29).

The proof is complete. □

Remark 2 *In the proof of the existence Theorem 1, the generator has the following quadratic growth:*

$$
|f^0(t, x, v, p, r)| \leq \lambda_0(t, x) + \lambda_1|v| + \lambda_2(|p|^2 + |r|^2).
$$

Assumption (**H**2) *gives that* $\lambda_0 \in \mathbb{H}^0(\mathbb{R}^d)$*. Therefore,* $\lambda_0 + \lambda_1|v| \in \mathbb{H}^0(\mathbb{R}^d)$ *and* $\lambda_2(|p|^2 + |r|^2) \in \mathbb{L}^1(\mathbb{R}^d)$*, yielding the required condition* $f^0 \in \mathbb{H}^0(\mathbb{R}^d) + \mathbb{L}^1(\mathbb{R}^d)$ *to apply Theorem 4. Similarly, in the proof of the uniqueness Theorem 2, the generator has the following quadratic growth:*

$$
|f^0(t, x, v, p, r)| \leq l_0(t, x) + l_1(|p|^2 + |r|^2).
$$

In view of Assumption (**H**6)*, we have* $f^0 \in \mathbb{L}^1(\mathbb{R}^d)$*, and thus Theorem 4 can be used.*

Remark 3 *In comparison with Lemma 2, Theorem 4 requires the extra condition* $\Phi(0) = 0$ *on* Φ*. Otherwise,* $\Phi(0) = c \neq 0$*, and a strictly positive deterministic function* $u(t, x)$ *of compact support and only depending on x obviously satisfies all the conditions required in Theorem 4, but the integral*

$$
\int_{\mathbb{R}^d \setminus supp\{u(t,\cdot)\}} \Phi(u(t, x))\, dx = \int_{\mathbb{R}^d \setminus supp\{u(t,\cdot)\}} c\, dx
$$

is not finite for each t and thus not integrable in t. If $\Phi(0) = 0$*, using the second-order Taylor's extension, we see that the spatial integral*

$$
\int_{\mathbb{R}^d} |\Phi(u(t, x))| dx \leq M \int_{\mathbb{R}^d} |u(t, x)|^2 dx
$$

is integrable in t *on* $[0, T]$. *Therefore, the extra condition* $\Phi(0) = 0$ *is crucial to validity of the Itô's formula stated in Theorem 4.*

We have the following type of Itô's formula for the case of $d = 1$.

Theorem 5 *Assume that the strictly positive function* $\zeta : [0, T] \to [\delta, +\infty)$ *satisfies the following ordinary differential equation:*

$$\zeta(t) = \zeta(T) + \int_t^T f(s)ds,$$

where $f \in L^\infty([0, T])$ *and the constant* $\delta > 0$. *The random functions* $f^i,\ f^0,\ q^k,\ u$ *are as defined as in Theorem 4, and the Eq. (17) holds for* $\forall\ \eta \in C_c^\infty(\mathbb{R}^d)$. *If* $d = 1$, *then for any* $\Phi \in C_b^2(\mathbb{R})$ *such that* $\Phi(x)\big|_{x\le 0} = 0$ *and* $|\Phi''(x)| \le K|x|$ *holds for some positive constant K, we have (noting that* $f = f^1$*)*

$$\begin{aligned}&\int_{\mathbb{R}^d} \Phi(u(t,x) - \zeta(t))dx - \int_{\mathbb{R}^d} \Phi(u(T,x) - \zeta(T))dx \\ &= \int_t^T \int_{\mathbb{R}^d} \left[\Phi'(u-\zeta)(f^0 - f) - \Phi''(u-\zeta)u_x f - \frac{1}{2}\Phi''(u-\zeta)|q|^2\right] dxds \\ &\quad - \int_t^T \int_{\mathbb{R}^d} \Phi'(u(s,x) - \zeta(s))q^k(s,x)dxdW_s^k, \quad a.s.. \end{aligned} \tag{34}$$

Proof Since $\Phi \in C_b^2(\mathbb{R})$, we have $|\Phi| + |\Phi'| + |\Phi''| \le M$. On the other hand, since $\Phi(x)\big|_{x\le 0} = 0$, we have

$$\Phi'(x)\Big|_{x\le 0} = \Phi''(x)\Big|_{x\le 0} = 0.$$

If $y \ge 0$, we have

$$|\Phi''(x-y)| \le K|x-y|\chi_{\{x\ge y\}} \le K|x|. \tag{35}$$

Since

$$\theta(x) := \mathbb{E}\int_0^T |u(s,x)f(s,x)|ds$$

is integrable over $\mathbb{R}$, we can choose $k(n)$ such that

$$\mathbb{E}\int_0^T \int_{|x|\ge k(n)} |uf|dxds \le \frac{1}{n^2}, \tag{36}$$

and $k(n)$ strictly increases to $+\infty$ as n tends to ∞. Construct the deterministic truncating function $\alpha_n(\cdot) \in C_c^\infty(\mathbb{R})$ as a mollified version of the following function

$$\alpha_n^0(x) := \begin{cases} 1, & |x| \le k(n); \\ 1 - \frac{1}{n}[|x| - k(n)], & |x| \in (k(n), k(n) + \frac{1}{n}); \\ 0, & |x| \ge k(n) + \frac{1}{n}. \end{cases}$$

We have $|\alpha_n'(x)| \leq 2n$ and $0 \leq \alpha_n(x) \leq 1$. Define

$$D^n := \{x \mid \alpha_n(x) = 1\}, \quad E^n := \{x \mid \alpha_n(x) = 0\}.$$

Obviously,

$$\{\alpha_n(x)\zeta(t), (t,x) \in [0,T] \times \mathbb{R}^d\} \in \mathbb{H}^1(\mathbb{R}) \cap L^2(\Omega,\ C([0,T],\ L^2(\mathbb{R}))),$$
$$\{\alpha_n(x) f(t), (t,x) \in [0,T] \times \mathbb{R}^d\} \in \mathbb{H}^0(\mathbb{R}) \cap \mathbb{L}^1(\mathbb{R}).$$

Moreover, for any $\eta \in C_c^\infty(\mathbb{R})$, we have that almost surely,

$$\begin{aligned}&\langle u(t,\cdot) - \alpha_n(\cdot)\zeta(t), \eta\rangle_{\mathbb{R}} - \langle u(T,\cdot) - \alpha_n(\cdot)\zeta(T), \eta\rangle_{\mathbb{R}} \\ &= \int_t^T \Big[\langle f^0(s,\cdot) - \alpha_n(\cdot) f(s), \eta\rangle_{\mathbb{R}} - \langle f(s,\cdot), \eta_x\rangle_{\mathbb{R}}\Big] ds - \int_t^T \langle q^k(s,\cdot), \eta\rangle_{\mathbb{R}}\, dW_s^k.\end{aligned}$$

Applying Theorem 4, we have

$$\begin{aligned}&\int_{\mathbb{R}} \Phi(u(t,x) - \alpha_n(x)\zeta(t))dx - \int_{\mathbb{R}} \Phi(u(T,x) - \alpha_n(x)\zeta(T))\,dx \\ &= \int_t^T\!\!\int_{\mathbb{R}} \Big[\Phi'(u-\alpha_n\zeta)(f^0-\alpha_n f) - \Phi''(u-\alpha_n\zeta)(u_x-(\alpha_n)_x\zeta) f - \frac{1}{2}\Phi''(u-\alpha_n\zeta)|q|^2\Big]\,dxds \\ &\quad - \int_t^T\!\!\int_{\mathbb{R}} \Phi'(u(s,x) - \alpha_n(x)\zeta(s))q^k(s,x)\,dxdW_s^k.\end{aligned} \tag{37}$$

In what follows, we now prove that as $n \to \infty$, all terms of the Eq. (37) converge to their respective counterparts of the Eq. (34).

First, we prove the following convergence

$$\int_{\mathbb{R}} \Phi(u(t,x) - \alpha_n(x)\zeta(t))dx \xrightarrow{a.s.} \int_{\mathbb{R}} \Phi(u(t,x) - \zeta(t))dx. \tag{38}$$

Note that

$$\begin{aligned}&\int_{\mathbb{R}} \Phi(u(t,x) - \alpha_n(x)\zeta(t))dx \\ &= \int_{D^n} \Phi(u-\zeta)dx + \int_{E^n} \Phi(u)dx + \int_{\mathbb{R}\setminus(D^n\cup E^n)} \Phi(u-\alpha_n\zeta)dx.\end{aligned}$$

Define the set $B_t := \{x \mid u(t,x) \geq \zeta(t)\}$. We have

$$\begin{aligned}&\left|\int_{\mathbb{R}} \Phi(u(t,x) - \alpha_n(x)\zeta(t))dx - \int_{\mathbb{R}} \Phi(u(t,x) - \zeta(t))dx\right| \\ &\leq \int_{\mathbb{R}\setminus(D^n\cup E^n)} |\Phi(u-\alpha_n\zeta) - \Phi(u-\zeta)|\,dx + \int_{E^n} |\Phi(u) - \Phi(u-\zeta)|\,dx \\ &\leq 2M\int_{\mathbb{R}\setminus(D^n\cup E^n)} dx + \int_{E^n\cap B_t} |\Phi(u) - \Phi(u-\zeta)|\,dx + \int_{E^n\setminus B_t} |\Phi(u)|\,dx \\ &\leq \frac{4M}{n} + \int_{E^n\cap B_t} |\Phi(u) - \Phi(u-\zeta)|\,dx + M\int_{E^n\setminus B_t} |u|^2 dx \\ &\leq \frac{4M}{n} + 2M\int_{E^n\cap B_t} dx + M\int_{E^n} |u|^2 dx.\end{aligned} \tag{39}$$

In view of Chebyshev's inequality, we have $\text{meas}(B_t) < +\infty$. Noting that $E^n = \{x \mid |x| \geq k(n) + \frac{1}{n}\}$, we have

$$\lim_{n\to+\infty}\int_{E^n}|u|^2dx = 0, \quad \lim_{n\to+\infty}\int_{E^n\cap B_t}dx \to 0,$$

and thus each side of inequality (39) converges to 0. Then, we have (38).

Now, consider both integrals of the right side of Eq. (37). For fixed (t, x), as $n \to \infty$, we have

$$\Phi''(u-\alpha_n\zeta)|q|^2 \to \Phi''(u-\zeta)|q|^2, \quad \Phi''(u-\alpha_n\zeta)fu_x \to \Phi''(u-\zeta)fu_x,$$

and they are dominated by $M|q|^2$ and $|fu_x|$, respectively. Using the dominated convergence theorem, we see that both integrals converge to their respective counterparts of Eq. (34).

Now, we prove the following convergence:

$$\int_t^T\int_{\mathbb{R}}\Phi'(u-\alpha_n\zeta)f\alpha_n dxds \xrightarrow{a.s.} \int_t^T\int_{\mathbb{R}}\Phi'(u-\zeta)fdxds. \tag{40}$$

Assume that $\|f\|_{L^\infty([0,T])} \leq N$. We have

$$\begin{aligned}
&\int_t^T\int_{\mathbb{R}}\left|\Phi'(u(s,x)-\alpha_n(x)\zeta(s))f(s)\alpha_n(x)-\Phi'(u(s,x)-\alpha_n(x)\zeta(s))f(s)\right|dxds\\
=&\int_t^T\int_{\mathbb{R}\setminus(D^n\cup E^n)}|f|\left|\Phi'(u-\alpha_n\zeta)\alpha_n-\Phi'(u-\zeta)\right|dxds+\int_t^T\int_{E^n}\left|\Phi'(u-\zeta)f\right|dxds\\
\leq&2M\int_t^T\int_{\mathbb{R}\setminus(D^n\cup E^n)}|f|dxds+\int_t^T\int_{E^n\cap B_s}\left|\Phi'(u-\zeta)f\right|dxds\\
\leq&2M\int_t^T\int_{[\mathbb{R}\setminus(D^n\cup E^n)]\cup(E^n\cap B_s)}|f|dxds\\
\leq&2MN\int_t^T\text{meas}\left\{[\mathbb{R}\setminus(D^n\cup E^n)]\cup(E^n\cap B_s)\right\}ds\\
\leq&2MN\left\{T\ \text{meas}\left[\mathbb{R}\setminus(D^n\cup E^n)\right]+\int_t^T\text{meas}(E^n\cap B_s)ds\right\}\\
\leq&\frac{4MNT}{n}+2MNT\int_t^T\text{meas}(E^n\cap B_s)ds.
\end{aligned} \tag{41}$$

First, $\lim_{n\to\infty}\text{meas}(E^n\cap B_s) = 0$ for any s. It follows from Chebyshev's inequality that

$$\text{meas}(B_s) \leq \frac{1}{\zeta^2(s)}\int_{\mathbb{R}}u^2(s,x)dx \leq \frac{1}{\delta^2}\int_{\mathbb{R}}u^2(s,x)dx.$$

Therefore, the function $\{\text{meas}(E^n\cap B_s), s\in[0,T]\}$ is dominated by

$$\frac{1}{\delta^2}\int_{\mathbb{R}}u^2(s,x)dx.$$

Using the dominated convergence theorem, the second term of the right side of inequality (41) converges to 0. Then, we have the convergence (40).

Now, we prove

$$\int_t^T \int_{\mathbb{R}} \Phi'(u-\alpha_n\zeta) f^0 dxds \xrightarrow{a.s.} \int_t^T \int_{\mathbb{R}} \Phi'(u-\zeta) f^0 dxds. \tag{42}$$

Consider both cases of $f^0 \in \mathbb{L}^1(\mathbb{R})$ and $f^0 \in \mathbb{H}^0(\mathbb{R})$.

(i) For $f^0 \in \mathbb{L}^1(\mathbb{R}^d)$, fixing (t, x), we have

$$\lim_{n\to\infty} \Phi'(u-\alpha_n\zeta) f^0 = \Phi'(u-\zeta) f^0,$$

and $|\Phi'(u-\alpha_n\zeta) f^0| \le M|f^0|$. Using the dominated convergence theorem, we have the convergence (42).

(ii) For $f^0 \in \mathbb{H}^0(\mathbb{R}^d)$, using Taylor's extension, we have

$$\begin{aligned}
&\int_t^T \int_{\mathbb{R}} \left|\Phi'(u-\alpha_n\zeta) f^0 - \Phi'(u-\alpha_n\zeta) f^0\right| dxds \\
=&\int_t^T \int_{\mathbb{R}\setminus(D^n\cup E^n)} |f^0| \left|\Phi'(u-\alpha_n\zeta) - \Phi'(u-\zeta)\right| dxds \\
&+\int_t^T \int_{E^n\cap B_s} |f^0| \left|\Phi'(u) - \Phi'(u-\zeta)\right| dxds + \int_t^T \int_{E^n\setminus B_s} \left|\Phi'(u) f^0\right| dxds \\
\le& 2M \int_t^T \int_{[\mathbb{R}\setminus(D^n\cup E^n)]\cup(E^n\cap B_s)} |f^0| dxds + M \int_t^T \int_{E^n} \left|uf^0\right| dxds.
\end{aligned} \tag{43}$$

It is sufficient to show that the first integral of the right side of the last inequality converges to 0 as n tends to ∞. Using Hölder's inequality, we have

$$\begin{aligned}
&\int_t^T \int_{[\mathbb{R}\setminus(D^n\cup E^n)]\cup(E^n\cap B_s)} |f^0| dxds \\
\le& \int_t^T \left\{\text{meas}\left\{[\mathbb{R}\setminus(D^n\cup E^n)]\cup(E^n\cap B_s)\right\}\right\}^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f^0|^2 dx\right)^{\frac{1}{2}} ds \\
\le& \left\{\int_t^T \text{meas}\left\{[\mathbb{R}\setminus(D^n\cup E^n)]\cup(E^n\cap B_s)\right\} ds\right\}^{\frac{1}{2}} \left(\int_t^T \int_{\mathbb{R}} |f^0|^2 dxds\right)^{\frac{1}{2}}.
\end{aligned}$$

Proceeding identically as before, in the right side of the last inequality, the first factor converges to 0, and the second factor is bounded. Therefore, the right side of the last inequality (43) converges to 0. In summary, as $f^0 \in \mathbb{H}^0(\mathbb{R}^d) + \mathbb{L}^1(\mathbb{R}^d)$, we have the convergence (42).

Now, we prove the following convergence

$$\int_t^T \int_{\mathbb{R}} f\Phi''(u-\alpha_n\zeta)(\alpha_n)_x \zeta dxds \xrightarrow{a.s.} 0 \tag{44}$$

for some subsequence $n := n_i$, which tends to $+\infty$ as $i \to +\infty$. In view of the increasing property of the upper bound Kx^+ of the function $|\Phi''(x)|$, and inequalities (35), (36), and $|\alpha_n'(x)| \le 2n$, we have

$$\begin{aligned} &\mathbb{E}\int_t^T \int_{\mathbb{R}} \left| f\Phi''(u-\alpha_n\zeta)(\alpha_n)_x\zeta \right| dxds \\ \le&(|\zeta(T)|+TN)\,\mathbb{E}\int_t^T \int_{\mathbb{R}\setminus(D^n\cup E^n)} 2n|\Phi''(u-\alpha_n\zeta)f|dxds \\ \le&2K(|\zeta(T)|+TN)\,\mathbb{E}\int_t^T \int_{\mathbb{R}\setminus(D^n\cup E^n)} n|uf|dxds \\ \le&\frac{2K(|\zeta(T)|+TN)}{n} \to 0, \quad \text{as} n\to+\infty, \end{aligned} \tag{45}$$

which yields (44).

Finally, we prove the convergence of the stochastic integral in the right side of Eq. (37). Identically as in the proof of Theorem 4, it is sufficient to prove the following convergence

$$\lim_{n\to+\infty} \mathbb{E}\sqrt{\int_t^T \left\{\int_{\mathbb{R}^d} [\Phi'(u-\alpha_n\zeta)-\Phi'(u-\zeta)]\,q^k dx\right\}^2 ds} = 0. \tag{46}$$

For any k, we have

$$\begin{aligned} &\left\{\int_{\mathbb{R}} \left[\Phi'(u-\alpha_n\zeta)-\Phi'(u-\zeta)\right] q^k dx\right\}^2 \\ \le&\left(2M\int_{[\mathbb{R}\setminus(D^n\cup E^n)]\cup(E^n\cap B_s)} |q^k|dx\right)^2 + \left[\int_{E^n\setminus B_s} \left|\Phi'(u)q^k\right| dx\right]^2 \\ \le&\, 4M^2 \text{ meas}\left\{[\mathbb{R}\setminus(D^n\cup E^n)]\cup(E^n\cap B_s)\right\}\left(\int_{\mathbb{R}} |q^k|^2dx\right)^2 + M^2\left[\int_{E^n\setminus B_s} |uq^k|dx\right]^2. \end{aligned}$$

Using the same arguments as in (41), we see that the right side of the last inequality converges to 0 as n tends to $+\infty$. Since $\zeta > 0$, we have

$$|\Phi'(u-\zeta)| \le M|u-\zeta|\chi_{\{u\ge\zeta\}} \le M|u|.$$

Using the same arguments as in (33) in the proof of Theorem 4, we arrive at the convergence (46) and see that the stochastic integral in Eq. (34) has zero expectation.

The proof is complete. □

Remark 4 *In the preceding proof, due to the assumption that the spatial dimension $d = 1$, the truncating function α_n has the following useful properties: (i) the measure of the set $\mathbb{R}^d \setminus (D^n \cup E^n)$ tends to 0, and (ii) $|\alpha_n'(x)| \le 2n$. In the case where $d \ge 2$, for the same truncating function α_n, the preceding property (ii) and thus inequality (45) are still true. However, since*

$$meas\left[\mathbb{R}^d\setminus(D^n\cup E^n)\right] = O\left(\left[k(n)+\frac{1}{n}\right]^d - [k(n)]^d\right) = O\left(\frac{[k(n)]^{d-1}}{n}\right),$$

it is not clear whether there is some $k(n)$ such that the preceding sequence of measures converges to 0 as $n\to\infty$ and meanwhile inequality (36) is guaranteed.

3.3 The proof of the main results

In this subsection, we prove Theorems 1 and 2. Once the suitable Itô's formula can be applied, all the arguments to prove the existence and uniqueness are quite natural. We mainly follow the proof of Du and Chen (2012, Theorem 1.1), except that the spatial integrals over the whole space $\mathbb{R}^d$ of some quantities have to be carefully estimated to guarantee that they are finite.

First, we have the following a priori estimate.

Lemma 5 *Let Assumptions* (**H**2) − (**H**5) *be satisfied, and BSPE (1)–(2) has a weak solution* $(u, q) \in \mathbb{H}^1(\mathbb{R}^d) \times \mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$ *such that* $u \in \mathbb{L}^\infty(\mathbb{R}^d) \cap L^2(\Omega,\ C([0, T],\ L^2(\mathbb{R}^d)))$. *Then, we have*

$$\|u(t,\cdot)\|_{L^\infty(\Omega\times\mathbb{R}^d)} \le \frac{e^{\lambda_1(T-t)}-1}{\lambda_1}\|\lambda_0\|_{\mathbb{L}^\infty(\mathbb{R}^d)} + e^{\lambda_1(T-t)}\|\varphi\|_{L^\infty(\Omega\times\mathbb{R}^d)}. \tag{47}$$

Further, there is a constant C which only depends on $\|\varphi\|_{L^2(\Omega\times\mathbb{R}^d)}$, $\|\lambda_0\|_{\mathbb{L}^2(\mathbb{R}^d)}$ *and constants* $\mu_0,\ \lambda_1, T$, *such that*

$$\|u_x\|^2_{\mathbb{H}^0(\mathbb{R}^d)} + \|q\|^2_{\mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})} \le C. \tag{48}$$

Proof We follow the routine in the proof of Du and Chen (2012, Proposition 5.1). First, to prove (47), we consider the following ordinary differential equation

$$\xi(t) = \|\varphi\|_{L^\infty(\Omega\times\mathbb{R}^d)} + \int_t^T (\lambda_1\xi(s) + \|\lambda_0\|_{\mathbb{L}^\infty(\mathbb{R}^d)})ds.$$

It has the following explicit solution

$$\xi(t) = \frac{e^{\lambda_1(T-t)}-1}{\lambda_1}\|\lambda_0\|_{\mathbb{L}^\infty(\mathbb{R}^d)} + e^{\lambda_1(T-t)}\|\varphi\|_{L^\infty(\Omega\times\mathbb{R}^d)}.$$

We construct function $\Phi_1 \in C^2(\mathbb{R})$ such that

$$\Phi_1(x) = \begin{cases} e^{2\lambda(M_1+1)} - [1 + 2\lambda(M_1+1) + 2\lambda^2(M_1+1)^2], & x \in [M_1+1, +\infty]; \\ e^{2\lambda x} - (1 + 2\lambda x + 2\lambda^2 x^2), & x \in [0, M_1]; \\ 0, & x \in (-\infty, 0], \end{cases}$$

where the positive constant

$$M_1 := \frac{e^{\lambda_1 T}}{\lambda_1}\|\lambda_0\|_{\mathbb{L}^\infty(\mathbb{R}^d)} + e^{\lambda_1 T}\|\varphi\|_{L^\infty(\Omega\times\mathbb{R}^d)} + \|u\|_{\mathbb{L}^\infty(\mathbb{R}^d)}.$$

It can be verified that both functions ξ and Φ_1 satisfy all the requirements of Theorem 5. We can apply Itô's formula (34) to $u(t, x) - \xi(t)$, and follow the proof of Du and Chen (2012, Proposition 5.1) to arrive at inequality (47).

Now, we use Itô's formula (18) to prove inequality (48). For this purpose, construct $\Phi_2 \in C^2(\mathbb{R})$ such that

$$\Phi_2(x) = \begin{cases} \frac{1}{2\lambda^2}[e^{2\lambda(M_2+1)} - 1 - 2\lambda(M_2+1)], & x \in [M_2+1, +\infty]; \\ \frac{1}{2\lambda^2}(e^{2\lambda x} - 1 - 2\lambda x), & x \in [0, M_2]; \\ \Phi_2(-x), & x \in (-\infty, 0], \end{cases}$$

where the positive constant $M_2 := \|u\|_{\mathbb{L}^\infty(\mathbb{R}^d)}$. It can be verified that the function Φ_2 satisfies all the requirements of Theorem 4. We can apply Itô's formula (18) to $u(t,x)$, and follow the proof of Du and Chen (2012, Proposition 5.1) to arrive at the following inequality

$$\sup_{t\in[0,T]} \mathbb{E}\int_{\mathbb{R}^d} |u|^2 dx \le \left(\frac{k_2}{k_1}\|\varphi\|_{L^2(\Omega\times\mathbb{R}^d)} + \frac{1}{2k_1}\|\lambda_0\|_{\mathbb{L}^2(\mathbb{R}^d)}\right) e^{\frac{k_4^2+2k_4\lambda_1}{2k_1}T}$$

for constants $k_1,\ k_2,\ k_3$, and k_4.

We have

$$\begin{aligned}
&\mu_0\mathbb{E}\int_t^T \int_{\mathbb{R}^d} (|u_x|^2 + |q|^2) dxds \\
&\le k_2\|\varphi\|_{L^2(\Omega\times\mathbb{R}^d)} + \frac{1}{2}\|\lambda_0\|_{\mathbb{L}^2(\mathbb{R}^d)} + (\frac{k_4^2}{2} + k_4\lambda_1)\int_t^T \mathbb{E}\int_{\mathbb{R}^d} |u|^2 dxds \\
&\le k_2\|\varphi\|_{L^2(\Omega\times\mathbb{R}^d)} + \frac{1}{2}\|\lambda_0\|_{\mathbb{L}^2(\mathbb{R}^d)} + (\frac{k_4^2}{2} + k_4\lambda_1)T \sup_{t\in[0,T]} \mathbb{E}\int_{\mathbb{R}^d} |u|^2 dxds \\
&\le \left[k_2 + (\frac{k_4^2}{2} + k_4\lambda_1)\frac{k_2 T}{k_1} e^{\frac{k_4^2+2k_4\lambda_1}{2k_1}T}\right] \|\varphi\|_{L^2(\Omega\times\mathbb{R}^d)} \\
&\quad + \left[\frac{1}{2} + (\frac{k_4^2}{2} + k_4\lambda_1)\frac{T}{2k_1} e^{\frac{k_4^2+2k_4\lambda_1}{2k_1}T}\right] \|\lambda_0\|_{\mathbb{L}^2(\mathbb{R}^d)}.
\end{aligned} \tag{49}$$

The proof is then complete. □

We have the following monotone convergence theorem for BSPEs of quadratic generators.

Lemma 6 *Assume that random functions $f^n(\omega,t,x,v,p,r)$ and $f(\omega,t,x,v,p,r)$ satisfy* (**H**1), *$\varphi^n(x)$ and $\varphi(x)$ satisfy* (**H**4), *and random coefficients a^{ij} and σ^{ik} satisfy* (**H**3). *Moreover, we make the following three assumptions.*

(i) For any (ω,t,x), f^n locally uniformly in (v,p,r) converges to f, and φ^n converges to φ strongly in $L^2(\Omega\times\mathbb{R}^d)$.

(ii) There are positive constants $\widetilde{\lambda}$ and nonnegative random function $\widetilde{\lambda_2} \in \mathbb{L}^\infty(\mathbb{R}^d)\cap\mathbb{L}^2(\mathbb{R}^d)$ such that

$$|f^n(t,x,v,p,r)| \le \widetilde{\lambda_2}(t,x) + \widetilde{\lambda}\mu_0(|p|^2 + |r|^2).$$

(iii) For any integer n, BSPE (f^n,φ^n) has a weak solution $(u^n,q^n)\in\mathbb{H}^1(\mathbb{R}^d)\times\mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$ such that $u^n \in \mathbb{L}^\infty(\mathbb{R}^d)\cap L^2(\Omega,\ C([0,T],\ L^2(\mathbb{R}^d)))$. Moreover, $\{u^n\}_n$ is monotone and is uniformly bounded.

Then, BSPE (1)–(2) has a weak solution $(u,q)\in\mathbb{H}^1(\mathbb{R}^d)\times\mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$ such that $u\in\mathbb{L}^\infty(\mathbb{R}^d)\cap L^2(\Omega,\ C([0,T],\ L^2(\mathbb{R}^d)))$, and

$$\lim_{n\to\infty}\|u^n - u\|_{\mathbb{H}^1(\mathbb{R}^d)} = 0, \quad \lim_{n\to\infty}\|q^n - q\|_{\mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})} = 0.$$

Proof The whole proof consists of the following three steps, as in Du and Chen (2012, Proposition 5.2). Step 1 is devoted to the proof of the strong convergence of (u^n, q^n) to (u, q) in the space $\mathbb{H}^1(\mathbb{R}^d) \times \mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$. Step 2 is devoted to proving that (u, q) is a weak solution to BSPE (1)−(2). Step 3 is devoted to the proof of the inclusion $u \in L^2(\Omega,\ C([0, T],\ L^2(\mathbb{R}^d)))$. Step 3 is the same as that of Du and Chen (2012, the proof of Proposition 5.2) and thus is omitted here. We only sketch the first two steps.

Step 1. From the assumptions, we see that there is a constant $M_3 > 0$ such that for any integers n, m, we have $\|u^n - u^m\|_{\mathbb{L}^\infty(\mathbb{R}^d)} \le M_3$. Define the function $\Phi_3 \in C^2(\mathbb{R})$ such that

$$\Phi_3(x) := \begin{cases} \frac{1}{200\widetilde{\lambda}^2}[e^{20\widetilde{\lambda}(M_3+1)} - 1 - 20\lambda(M_3 + 1)], & x \in [M_3 + 1, +\infty]; \\ \frac{1}{200\widetilde{\lambda}^2}(e^{2\widetilde{\lambda}x} - 1 - 20\lambda x), & x \in [0, M_3]; \\ \Phi_2(-x), & x \in (-\infty, 0]. \end{cases}$$

The function Φ_3 satisfies all requirements of Theorem 4. We then can apply Itô's formula (18) in Theorem 4 to $u^n - u^m$. Then, following the remaining arguments in Step 1 of Du and Chen (2012, the proof of Proposition 5.2), to show that (u^n, q^n) converges to (u, q) strongly in the space $\mathbb{H}^1(\mathbb{R}^d) \times \mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$ and $u \in \mathbb{L}^\infty(\mathbb{R}^d)$.

Step 2. In view of Lemma 3, we assume without loss of generality that all the three functions $\sup_n |u^n|$, $\sup_n |u^n_{x^j}|$, and $\sup_n |q^{k,n}|$ belong to the space $\mathbb{H}^0(\mathbb{R}^d)$, and each of $(u^n,\ u^n_{x^j}, q^{k,n})$ converges to $(u,\ u_{x^j}, q^k)$ almost everywhere in (ω, t, x). Our spatial integral over the whole space $\mathbb{R}^d$ brings difficulty in this step. We begin with the definition of weak solutions. Since (u^n, q^n) is a weak solution to BSPE (f^n, φ^n), we have for any $\eta \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned} &\int_{\mathbb{R}^d} u^n(t, x)\eta(x)dx - \int_{\mathbb{R}^d} \varphi^n(x)dx \\ &= \int_t^T \int_{\mathbb{R}^d} \left[-(a^{ij} u^n_{x^j} + \sigma^{ik} q^{k,n})\eta_{x^j} + f^n\left(s, x, u^n, u^n_x, q^n\right)\eta\right] dxds \\ &\quad - \int_t^T \int_{\mathbb{R}^d} q^{k,n}(s, x)\eta(x)dxdW^k_s, \quad a.s. \end{aligned} \tag{50}$$

For fixed η, take $C_\eta := \mathrm{supp}(\eta)$, and positive constants $M_4 := \|\eta\|_{W^{1,+\infty}(\mathbb{R}^d)}$ and $M_5 := \sup_{i,j} |a^{ij}| + \sup_{i,k} |\sigma^{ik}|$. It is sufficient to prove the convergence of every integral in Eq. (50) .

First, since

$$\int_{\mathbb{R}^d} \left|[u^n(t, x) - u(t, x)]\eta(x)\right| dx = \int_{C_\eta} \left|[u^n(t, x) - u(t, x)]\eta(x)\right| dx,$$

noting that $|[u^n(t, x) - u(t, x)]\eta(x)| \le 2M_4 \sup_n |u^n(t, x)|$ and the right side of the last inequality is integrable over the bounded domain C_η, we have from the dominated convergence theorem that

$$\int_{\mathbb{R}^d} \left|[u^n(t, x) - u(t, x)]\eta(x)\right| dx \xrightarrow{a.s.} 0.$$

Identically, noting that the sequence $\{\varphi^n\}_n$ is uniformly bounded, we can also prove

$$\int_{\mathbb{R}^d} \left|[\varphi^n(x) - \varphi(x)]\eta(x)\right| dx \xrightarrow{a.s.} 0.$$

Since

$$\begin{aligned}
&\int_t^T \int_{\mathbb{R}^d} \left|-[a^{ij}u^n_{x^j} + \sigma^{ik}q^{k,n}]\eta_{x^j}\right| dxds \\
\leq &\int_t^T \int_{\mathbb{R}^d} 2M_4M_5 \left(\sup_n |u^n_{x^j}| + \sup_n |q^{k,n}|\right) \chi_{\{x\in C_\eta\}} dxds \\
\leq &4M_4M_5 \left(\int_t^T \int_{\mathbb{R}^d} \left(\sup_n |u^n_{x^j}|^2 + \sup_n |q^{k,n}|^2\right) dxds\right)^{\frac{1}{2}} \left(\int_t^T \int_{\mathbb{R}^d} \chi_{\{x\in C_\eta\}} dxds\right)^{\frac{1}{2}}.
\end{aligned}$$

We can see that the dominating function $\left(\sup_n |u^n_{x^j}| + \sup_n |q^{k,n}|\right) \chi_{\{x\in C_\eta\}}$ has the necessary integrability required in the dominated convergence theorem, and we have

$$\int_t^T \int_{\mathbb{R}^d} \left|-(a^{ij}u^n_{x^j} + \sigma^{ik}q^{k,n})\eta_{x^j}\right| dxds \xrightarrow{a.s.} \int_t^T \int_{\mathbb{R}^d} \left|-(a^{ij}u_{x^j} + \sigma^{ik}q^{k})\eta_{x^j}\right| dxds.$$

From Assumption (i), we have the following: almost everywhere at (ω, t, x),

$$\lim_{n\to\infty} f^n(t, x, u^n(t, x), u^n_x(t, x), q^n(t, x)) = f(t, x, u(t, x), u_x(t, x), q(t, x)).$$

Since

$$\begin{aligned}
&\int_t^T \int_{\mathbb{R}^d} \left|f^n(t, x, u^n(t, x), u^n_x(t, x), q^n(t, x))\eta(x)\right| dxds \\
\leq &\int_t^T \int_{\mathbb{R}^d} \left[\widetilde{\lambda}_2(t, x) + \widetilde{\lambda}\mu_0 \sup_n(|u^n_x|^2 + |q^n|^2)\right] \chi_{\{x\in C_\eta\}}\, dxds \\
\leq &\int_t^T \int_{\mathbb{R}^d} \widetilde{\lambda}\mu_0 \sup_n(|u^n_x|^2 + |q^n|^2)\, dxds + \int_t^T \int_{C_\eta} \widetilde{\lambda}_2(t, x)\, dxds \\
\leq &\int_t^T \int_{\mathbb{R}^d} \widetilde{\lambda}\mu_0 \sup_n(|u^n_x|^2 + |q^n|^2)\, dxds + T\ \mathrm{meas}(C_\eta)\ \|\widetilde{\lambda}_2\|_{\mathbb{L}^\infty(\mathbb{R}^d)},
\end{aligned}$$

the dominant function $\left[\widetilde{\lambda}_2(t, x) + \widetilde{\lambda}\mu_0 \sup_n(|u^n_x|^2 + |q^n|^2)\right] \chi_{\{x\in C_\eta\}}$ has the integrability required in the dominant convergence theorem, and we have

$$\int_t^T\int_{\mathbb{R}^d} \left|f^n(t, x, u^n, u^n_x, q^n)\eta\right| dxds \xrightarrow{a.s.} \int_t^T\int_{\mathbb{R}^d} |f(t, x, u, u_x, q)\eta|\, dxds. \quad (51)$$

Finally, we prove for $k = 1, 2, \cdots, d_0$ that

$$\mathbb{E}\int_t^T \left\{\int_{\mathbb{R}^d} \left[q^{k,n}(t, x) - q^k(t, x)\right]\eta(x)dx\right\}^2 ds \to 0. \quad (52)$$

Since $[q^{k,n}(t,x) - q^k(t,x)]\eta(x) \to 0$ almost everywhere in (ω, t, x) and

$$\begin{aligned}
&\mathbb{E}\int_t^T \left\{\int_{\mathbb{R}^d}\left[q^{k,n}(t,x) - q^k(t,x)\right]\eta(x)dx\right\}^2 ds \\
\leq &4M_4^2\, \mathbb{E}\int_t^T \left[\int_{\mathbb{R}^d}\sup_n |q^{k,n}|\ \chi_{\{x\in C_\eta\}}dx\right]^2 ds \\
\leq &4M_4^2\ \mathrm{meas}(C_\eta)\, \mathbb{E}\int_t^T\int_{\mathbb{R}^d}\sup_n |q^{k,n}|^2 dxds,
\end{aligned}$$

apply twice the dominant convergence theorem, and we have (52).

Concluding the above and setting $n \to \infty$ in Eq. (50), we see that (u, q) is a weak solution to BSPE (1)–(2). The proof is complete. □

Remark 5 *The last lemma does not invoke Theorem 5, and is true for any dimension d.*

The following gives a weak solution of a quadratic BSPE via that of the exponential-transferred one.

Lemma 7 *Assume that* $(v, r) \in \mathbb{H}^1(\mathbb{R}^d) \times \mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$ *is a weak solution of BSPE*

$$\begin{cases} dv = -\left[(a^{ij}v_{x^j} + \sigma^{ik}r^k)_{x^i} + F(t,x)\right]dt + r^k dW_t^k, \\ v(T,x) = e^{2\lambda\varphi(x)} - 1, \end{cases}$$

where both functions $F(\cdot,\cdot)$ *and* $\varphi(\cdot)$ *are bounded, and that there are two constants* $\gamma > 0$ *and* $\Gamma > 0$ *such that* $0 \leq \gamma \leq v + 1 \leq \Gamma$. *Then, the pair* (u, q) *defined by*

$$u := \frac{1}{2\lambda}\ln(v+1), \quad q := \frac{r}{2\lambda(v+1)},$$

is a weak solution of BSPE

$$\begin{cases} du = -\left[(a^{ij}u_{x^j} + \sigma^{ik}q^k)_{x^i} + f(t,x,u,u_x,q)\right]dt + q^k dW_t^k, \\ u(T,x) = \varphi(x), \end{cases}$$

where

$$f(t,x,u,u_x,q) := \frac{1}{2\lambda}e^{-2\lambda u}F(t,x) + \lambda(2a^{ij}u_{x^i}u_{x^j} + 2\sigma^{ik}u_{x^i}q^k + |q|^2).$$

Proof From the definition of (u, q), we have

$$|u| \leq \frac{\gamma + |\ln\gamma|}{2\lambda\gamma}|v|, \quad |u_{x^i}| = \frac{|v_{x^i}|}{2\lambda(v+1)} \leq \frac{|v_{x^i}|}{2\lambda\gamma}, \quad |q| \leq \frac{|r|}{2\lambda\gamma}.$$

Therefore, $(u, q) \in \mathbb{H}^1(\mathbb{R}^d) \times \mathbb{H}^0(\mathbb{R}^d;\ \mathbb{R}^{d_0})$.

Now, we can follow the same arguments of Du and Chen (2012, Lemma 3.5) to complete the proof. □

At last, we prove the existence and uniqueness of weak solutions. Now, we first prove the existence Theorem 1 of weak solutions.

Proof Define

$$M_6 := \frac{e^{\lambda_1(T-t)}-1}{\lambda_1}\|\lambda_0\|_{\mathbb{L}^\infty(\mathbb{R}^d)} + e^{\lambda_1(T-t)}\|\varphi\|_{L^\infty(\Omega\times\mathbb{R}^d)}.$$

From Lemma 5, we see that if (u, q) is a weak solution of BSPE (1)$-$(2) with u being bounded, then $u \le M_6$.

Identically as in the proof of Du and Chen (2012, Theorem 2.1) (see Du and Chen (2012, pages 459 and 460)), we construct the generator F^n and consider the associated BSPE:

$$\begin{cases} dv^n = -\Big[(a^{ij}v^n_{x^j} + \sigma^{ik}r^{n,k})_{x^i} + F^n(t, x, v^n, v^n_x, r^n)\Big]dt + r^{n,k}dW^k_t, \\ v^n(T, x) = e^{2\lambda\varphi(x)} - 1. \end{cases}$$

In view of Du and Tang (2010, Lemma 2.3), the last BSPE has a weak solution $(v^n, r^n) \in \mathbb{H}^1(\mathbb{R}) \cap L^2(\Omega,\ C([0, T],\ L^2(\mathbb{R}^d))) \times \mathbb{H}^0(\mathbb{R};\ \mathbb{R}^{d_0})$ such that

$$e^{-2\lambda(M_6+1)} - 1 \le v^{n+1} \le v^n \le e^{2\lambda(M_6+1)} - 1.$$

Define

$$u^n := \frac{1}{2\lambda}\ln(v^n + 1), \quad q^n := \frac{r^n}{2\lambda(v^n + 1)}.$$

Lemma 7 indicates that (u^n, q^n) is a weak solution to BSPE:

$$\begin{cases} du^n = -\Big[(a^{ij}u^n_{x^j} + \sigma^{ik}q^{n,k})_{x^i} + f^n(t, x, u^n, u^n_x, q^n)\Big]dt + q^{n,k}dW^k_t, \\ u^n(T, x) = \varphi(x) \end{cases}$$

where the generator

$$\begin{aligned} f^n(t, x, u, u_x, q) :=& \frac{1}{2\lambda}e^{-2\lambda u}F^n(t, x, e^{2\lambda u} - 1, 2\lambda e^{2\lambda u}u_x, 2\lambda e^{2\lambda u}q) \\ &+ \lambda(2a^{ij}u_{x^i}u_{x^j} + 2\sigma^{ik}u_{x^i}q^k + |q|^2). \end{aligned}$$

Noting that $\{u^n\}_n$ is monotone, and using Lemma 6, we see that the following BSPE

$$\begin{aligned} &du = -\big[(a^{ij}u_{x^j} + \sigma^{ik}q^k)_{x^i} + \widetilde{f}(t, x, u, u_x, q)\big]dt + q^k dW^k_t, \\ &u(T, x) = \varphi(x) \end{aligned}$$

for the limit $\widetilde{f}$ of $\{f^n\}_n$, has a weak solution (u, q) with u being the point-wise limit of u^n and belonging to the space $\mathbb{L}^\infty(\mathbb{R}) \cap L^2(\Omega,\ C([0, T],\ L^2(\mathbb{R})))$. While if $u \le M_6$, we have $\widetilde{f} = f$.

Concluding the above, we see that (u, q) is a weak solution to BSPE (1)$-$(2) such that $u \in \mathbb{L}^\infty(\mathbb{R})$. The proof is complete. □

Next, we prove the uniqueness Theorem 2 of weak solutions.

Proof Define $\widetilde{u}^+ := (u^1 - u^2)^+$ and $\widetilde{q} := q^1 - q^2$. Since both u^1 and u^2 are bounded, assume without loss of generality that $\widetilde{u}^+ \le M_7$. Construct a deterministic function $\Phi_4 \in C^2(\mathbb{R})$ such that

$$\Phi_4(x) = \begin{cases} (M_7+1)^{2m}, & x \in [M_7+1, +\infty]; \\ x^{2m}, & x \in [0, M_7]; \\ 0, & x \in (-\infty, 0], \end{cases}$$

where integer $m \geq 2$. It is easily verified that Φ_4 satisfies all the conditions of Theorem 4 such that Itô's formula (18) can be applied, and we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \left|\widetilde{u}^+(t,x)\right|^{2m} dx \\ &+m(2m-1)\int_t^T\int_{\mathbb{R}^d} \left(\widetilde{u}^+\right)^{2(m-1)} \left(2a^{ij}\widetilde{u}_{x^i}\widetilde{u}_{x^j} + 2\sigma^{ik}\widetilde{u}_{x^i}\widetilde{q}^k + |\widetilde{q}|^2\right)(s,x)\, dxds \\ = 2m&\int_t^T\int_{\mathbb{R}^d} \left(\widetilde{u}^+\right)^{2m-1} \widetilde{f}(s,x)\, dxds - 2m\int_t^T\int_{\mathbb{R}^d} \left(\widetilde{u}^+\right)^{2m-1} \widetilde{q}^k(s,x)\, dxdW_s^k \end{aligned} \quad (53)$$

where

$$\begin{aligned} \widetilde{f}(s,x) &:= f(s,x,(u^1,u^1_x,q^1)(s,x)) - f(s,x,(u^2,u^2_x,q^2)(s,x)) \\ &= \left(\int_0^1 f_u(s,x,\Lambda_\lambda(s,x))\, d\lambda\right)\widetilde{u}(s,x) \\ &\quad + \left(\int_0^1 f_z(s,x,\Lambda_\lambda(s,x))\, d\lambda\right)(\widetilde{u}_x,\widetilde{q})^*(s,x) \end{aligned} \quad (54)$$

with

$$\widehat{u}^\lambda := \lambda u^1 + (1-\lambda)u^2, \quad \widehat{q}^\lambda := \lambda q^1 + (1-\lambda)q^2, \quad \Lambda_\lambda(s,x) := \left(\widehat{u}^\lambda, \widehat{u}^\lambda_x, \widehat{q}^\lambda\right)(s,x).$$

The rest is the same as that of Du and Chen (2012) to complete the proof. □

Remark 6 *Our proof of Theorem 2 appeals to neither Theorem 5 nor the a priori estimate in Lemma 5, and therefore our uniqueness assertion applies to arbitrary spatial dimension d.*

Remark 7 *In contrast to Theorem 2 which requires that the quadratic generator satisfies (16), Du and Chen (2012, Theorem 2.1, page 450) actually requires the following weaker condition:*

$$|f(t,x,v,p,r)| \leq \widetilde{l}_0(t) + l_1(|p|^2 + |r|^2).$$

where $\widetilde{l}_0(\cdot) \in L^\infty([0,T])$. *This is because the latter paper* (Du and Chen 2012) *considers the Cauchy−Dirichlet problem in a bounded spatial domain, while we consider the Cauchy problem where the spatial domain is the whole space and is unbounded. Our assumption (16) ensures that the generator* $f \in \mathbb{L}^1(\mathbb{R}^d)$, *and then Theorem 4 can be applied.*

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